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PARAMETRIZED VARIATIONAL PRINCIPLES FOR MICROPOLAR ELASTICITY

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Abstract—A parametrized six-field variational principle for micropolar compressible linear elasticity is presented. The primary variables are symmetric and skew stresses, symmetric and skew strains, micropolar rotations and displacements. The governing functional is characterized by six free parameters. The connection between this formulation and the functionals with relaxed stress symmetry and independent rotations fields proposed by Reissner and Hughes–Brezzi for classical (non-polar) linear elasticity is examined. It is shown that the Hughes–Brezzi functionals are special cases of the parametrized functional but that the Reissner functionals are not. The former may be interpreted as a regularization (consistent stabilization) of the Reissner functionals that places them within the framework of micropolar elasticity. An eight-field parametrized principle that accounts for couple stresses is briefly described in the Appendix.

1. GOVERNING EQUATIONS

Consider a compressible linear micropolar body under static loading that occupies the volume V. The body is bounded by the surface S, with outward external normal n_i. The surface is decomposed into $S: S_d \cup S_t$. Displacements are prescribed on S_d while surface tractions are prescribed on S_t. Rectangular Cartesian coordinates will be used throughout. The four unknown volume fields are the displacement vector u_i, the infinitesimal strain tensor γ_{ij}, the stress tensor τ_{ij}, and the (antisymmetric) microrotation tensor θ_{ij}. The stress and strain tensors are not symmetric. The symmetric and antisymmetric parts of the stress tensor are σ_{ij} and s_{ij}, respectively. The symmetric and antisymmetric parts of the strain tensor are e_{ij} and ϕ_{ij}, respectively. The antisymmetric tensor of infinitesimal rotations (also called macrorotations) is ω_{ij}.

The problem data include: the body force field b_i per unit of volume in V, body couples c_i per unit volume in V, prescribed displacements $\hat{d}_i$ on S_d, and prescribed surface tractions $\hat{t}_i$ on S_t.

The governing field equations for an *isotropic* micropolar continuum without couple stresses are written below following Novacki (1970), with some notational changes. In the following equations, δ_{ij} is the Kronecker delta, ε_{ijk} denotes the permutator symbol ($\varepsilon_{ijk} = +1$ or -1 if i, j, k are distinct and form a positive or negative permutation, respectively, of 1, 2, 3; else $\varepsilon_{ijk} = 0$), λ and μ are the Lamé coefficients, and κ is a micropolar modulus that relates the antisymmetric tensors ϕ_{ij} and s_{ij}. In addition, a comma denotes the partial derivative with respect to the space coordinate whose index follows.

Strain-displacement and rotation-displacement equations in V:

$$\begin{aligned} \gamma_{ij} &= u_{j,i} - \theta_{ij} = e_{ij} + \omega_{ij} - \theta_{ij} = e_{ij} + \phi_{ij}, \\ \omega_{ij} &= \tfrac{1}{2}(u_{j,i} - u_{i,j}), \\ e_{ij} &= \tfrac{1}{2}(\gamma_{ij} + \gamma_{ji}) = \tfrac{1}{2}(u_{j,i} + u_{i,j}), \\ \phi_{ij} &= \tfrac{1}{2}(\gamma_{ij} - \gamma_{ji}) = \tfrac{1}{2}(u_{j,i} - u_{i,j}) - \theta_{ij} = \omega_{ij} - \theta_{ij}. \end{aligned} \tag{1}$$

Constitutive equations in V:

$$\begin{aligned} \tau_{ij} &= (\mu+\kappa)\gamma_{ij} + (\mu-\kappa)\gamma_{ji} + \lambda\delta_{ij}\gamma_{kk} = \sigma_{ij} + s_{ij}, \\ \sigma_{ij} &= \tfrac{1}{2}(\tau_{ij} + \tau_{ji}) = 2\mu e_{ij} + \lambda\delta_{ij} e_{kk}, \\ s_{ij} &= \tfrac{1}{2}(\tau_{ij} - \tau_{ji}) = 2\kappa\phi_{ij}. \end{aligned} \tag{2}$$

Equilibrium equations in V:

$$\tau_{ji,j}+b_i = \sigma_{ji,j}+s_{ji,j}+b_i = 0, \quad \varepsilon_{ijk}\tau_{jk}+c_i = 0. \tag{3}$$

Stress boundary conditions on S_t:

$$\tau_{ij}n_j = \hat{t}_i. \tag{4}$$

Displacement boundary conditions on S_d:

$$u_i = \hat{d}_i. \tag{5}$$

The foregoing equations apply if the presence of the couple stresses m_{ij} is neglected. The variational treatment is extended to that case in the Appendix.

For completeness, and to facilitate correlation with other references, eqns (1)–(5) are restated below in direct (index-free) tensor notation:

$$\left.\begin{aligned}
&\underline{\boldsymbol{\gamma}} = \nabla\mathbf{u}-\underline{\boldsymbol{\theta}} = \underline{\mathbf{e}}+\underline{\boldsymbol{\omega}}-\underline{\boldsymbol{\theta}} = \underline{\mathbf{e}}+\underline{\boldsymbol{\phi}},\\
&\underline{\boldsymbol{\omega}} = \tfrac{1}{2}(\nabla-\nabla^{T})\mathbf{u} = \text{skew}\,(\nabla\mathbf{u}),\\
&\underline{\mathbf{e}} = \tfrac{1}{2}(\nabla+\nabla^{T})\mathbf{u} = \text{symm}\,(\nabla\mathbf{u}) = \text{symm}\,\underline{\boldsymbol{\gamma}},\\
&\underline{\boldsymbol{\phi}} = \underline{\boldsymbol{\omega}}-\underline{\boldsymbol{\theta}} = \tfrac{1}{2}(\nabla-\nabla^{T})\mathbf{u}-\underline{\boldsymbol{\theta}} = \text{skew}\,(\nabla\mathbf{u}-\underline{\boldsymbol{\theta}}) = \text{skew}\,\underline{\boldsymbol{\gamma}},\\
&\underline{\boldsymbol{\tau}} = (\mu+\kappa)\underline{\boldsymbol{\gamma}}+(\mu-\kappa)\underline{\boldsymbol{\gamma}}^{T}+\lambda\underline{\mathbf{I}}\ \text{trace}\,\underline{\boldsymbol{\gamma}} = \underline{\boldsymbol{\sigma}}+\underline{\mathbf{s}},\\
&\underline{\boldsymbol{\sigma}} = \text{symm}\,\underline{\boldsymbol{\tau}} = 2\mu\underline{\mathbf{e}}+\lambda\underline{\mathbf{I}}\ \text{trace}\,\underline{\boldsymbol{\gamma}},\\
&\underline{\mathbf{s}} = \text{skew}\,\underline{\boldsymbol{\tau}} = 2\kappa\underline{\boldsymbol{\phi}},\\
&\text{div}\,\underline{\boldsymbol{\tau}}+\mathbf{b} = \text{div}\,(\underline{\boldsymbol{\sigma}}+\underline{\mathbf{s}})+\mathbf{b} = \mathbf{0},\\
&2\,\text{axial}\,\underline{\boldsymbol{\tau}}+\mathbf{c} = \mathbf{0},
\end{aligned}\right\}\ \text{in } V \tag{6}$$

$$\boldsymbol{\tau}_n = \hat{\mathbf{t}} \quad \text{on } S_t,$$

$$\mathbf{u} = \hat{\mathbf{d}} \quad \text{on } S_d.$$

Here an underlined bold symbol denotes a second order or higher tensor. This convention is used to distinguish tensors from their vector/matrix representations introduced in Section 2.1. No such distinction is needed for vectors such as **u**.

2. NOTATION

2.1. *Matrix notation*

To facilitate the construction and manipulation of variational matrix expressions, stresses and strains will be arranged as column vectors constructed from the respective tensors. The arrangement rules vary according to the symmetry properties and are best illustrated by specific examples.

For symmetric stress and strain tensors:

$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \equiv \boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix}, \quad \underline{\mathbf{e}} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{bmatrix} \equiv \mathbf{e} = \begin{Bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \end{Bmatrix}. \tag{7}$$

where $\sigma_{31} = \sigma_{13}$ and $e_{31} = e_{13}$. The factor of 2 in e maintains the equivalence of the stress–strain inner products; cf. (12) below.

For antisymmetric (skew) stress and strain tensors:

$$\underline{\mathbf{s}} = \begin{bmatrix} 0 & s_{12} & s_{13} \\ -s_{12} & 0 & s_{23} \\ -s_{13} & -s_{23} & 0 \end{bmatrix} \equiv \mathbf{s} = \begin{Bmatrix} s_{23} \\ s_{31} \\ s_{12} \end{Bmatrix}, \quad \underline{\boldsymbol{\phi}} = \begin{bmatrix} 0 & \phi_{12} & \phi_{13} \\ -\phi_{12} & 0 & \phi_{23} \\ -\phi_{13} & -\phi_{23} & 0 \end{bmatrix} \equiv \boldsymbol{\phi} = \begin{Bmatrix} 2\phi_{23} \\ 2\phi_{31} \\ 2\phi_{12} \end{Bmatrix}, \tag{8}$$

$$\underline{\boldsymbol{\theta}} = \begin{bmatrix} 0 & \theta_{12} & \theta_{13} \\ -\theta_{12} & 0 & \theta_{23} \\ -\theta_{13} & -\theta_{23} & 0 \end{bmatrix} \equiv \boldsymbol{\theta} = \begin{Bmatrix} 2\theta_{23} \\ 2\theta_{31} \\ 2\theta_{12} \end{Bmatrix}, \quad \underline{\boldsymbol{\omega}} = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix} \equiv \boldsymbol{\omega} = \begin{Bmatrix} 2\omega_{23} \\ 2\omega_{31} \\ 2\omega_{12} \end{Bmatrix}, \tag{9}$$

where $s_{31} = -s_{13}$ and $\phi_{31} = -\phi_{13}$. The factor of 2 applies only to kinematic skew (rotational) tensors, and again maintains inner product equivalence; cf. (12) below.

For general (unsymmetric) stress and strain tensors:

$$\underline{\boldsymbol{\tau}} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \equiv \boldsymbol{\tau} = \begin{Bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \\ \tau_{32} \\ \tau_{13} \\ \tau_{21} \end{Bmatrix}, \quad \underline{\boldsymbol{\gamma}} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} \equiv \boldsymbol{\gamma} = \begin{Bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \\ \gamma_{32} \\ \gamma_{13} \\ \gamma_{21} \end{Bmatrix}. \tag{10}$$

With these conventions operations between tensors of equal type can be easily translated to matrix form. For example, the inner products

$$\underline{\boldsymbol{\sigma}} : \underline{\mathbf{e}} = \sigma_{ij} e_{ij} = \boldsymbol{\sigma}^{\mathrm{T}} \mathbf{e}, \quad \underline{\mathbf{s}} : \underline{\boldsymbol{\phi}} = s_{ij} \phi_{ij} = \mathbf{s}^{\mathrm{T}} \boldsymbol{\phi}. \tag{11}$$

Problems arise, however, in combining different types. For example, $\boldsymbol{\tau} = \boldsymbol{\sigma} + \mathbf{s}$ is an inconsistent matrix operation because vectors $\boldsymbol{\sigma}$ and s have different dimensions. This difficulty can be circumvented by introducing "uncompressed" versions, in which components of symmetric and skew tensors are arranged as general tensors:

$$
{}^*\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix}, \quad {}^*\mathbf{s} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ s_{23} \\ s_{31} \\ s_{12} \\ -s_{23} \\ -s_{31} \\ -s_{12} \end{Bmatrix}, \quad {}^*\mathbf{e} = \begin{Bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \\ e_{23} \\ e_{31} \\ e_{12} \end{Bmatrix}, \quad {}^*\boldsymbol{\phi} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \phi_{23} \\ \phi_{31} \\ \phi_{12} \\ -\phi_{23} \\ -\phi_{31} \\ -\phi_{12} \end{Bmatrix}. \tag{12}
$$

Furthermore, $\boldsymbol{\tau} = {}^*\boldsymbol{\tau}$ and $\boldsymbol{\gamma} = {}^*\boldsymbol{\gamma}$; thus no distinction is needed there. This convention will let us consistently expand expressions such as the inner product of total stresses and strains:

$$
\tau_{ij}\gamma_{ij} = \boldsymbol{\tau}^T\boldsymbol{\gamma} = ({}^*\boldsymbol{\sigma}+{}^*\mathbf{s})^T({}^*\mathbf{e}+{}^*\boldsymbol{\phi}) = \boldsymbol{\sigma}^T\mathbf{e}+\mathbf{s}^T\boldsymbol{\phi}. \tag{13}
$$

2.2. *Matrix form of governing equations*

Using the matrix notation of Section 2.1, field equations (1)–(3) may be represented as follows:

Strain-displacement equations:

$$
\boldsymbol{\gamma} = {}^*\mathbf{e}+{}^*\boldsymbol{\phi}, \quad \mathbf{e} = \mathbf{D}\mathbf{u}, \quad \boldsymbol{\phi} = \boldsymbol{\omega}-\boldsymbol{\theta} = \mathbf{R}\mathbf{u}-\boldsymbol{\theta}. \tag{14}
$$

Constitutive equations:

$$
\boldsymbol{\tau} = {}^*\boldsymbol{\sigma}+{}^*\mathbf{s}, \quad \boldsymbol{\sigma} = \mathbf{E}\mathbf{e}, \quad \mathbf{s} = \mathbf{G}\boldsymbol{\phi}. \tag{15}
$$

Equilibrium equations:

$$
\mathbf{D}^T\boldsymbol{\sigma}+\mathbf{R}^T\mathbf{s}+\mathbf{b} = \mathbf{0}, \quad 2\mathbf{s}+\mathbf{c} = \mathbf{0}. \tag{16}
$$

In the above equations,

$$
\mathbf{D} = \begin{bmatrix} \partial/\partial x_1 & 0 & 0 \\ 0 & \partial/\partial x_2 & 0 \\ 0 & 0 & \partial/\partial x_3 \\ \partial/\partial x_2 & \partial/\partial x_1 & 0 \\ 0 & \partial/\partial x_3 & \partial/\partial x_2 \\ \partial/\partial x_3 & 0 & \partial/\partial x_1 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} -\partial/\partial x_2 & \partial/\partial x_1 & 0 \\ 0 & -\partial/\partial x_3 & \partial/\partial x_2 \\ \partial/\partial x_3 & 0 & -\partial/\partial x_1 \end{bmatrix} \tag{17}
$$

are the symmetric gradient and curl operators, respectively, in matrix form, and

$$
\mathbf{E} = \begin{bmatrix} \lambda+2\mu & \mu & \mu & 0 & 0 & 0 \\ \mu & \lambda+2\mu & \mu & 0 & 0 & 0 \\ \mu & \mu & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}, \quad \mathbf{G} = \kappa\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{18}
$$

In the sequel $\mathbf{E}$ and $\mathbf{G}$ are not restricted to these isotropic forms but can be *arbitrary non-singular* symmetric matrices. This allows anisotropy in the constitutive equations, subjected however to the restriction that the pairs $(\boldsymbol{\sigma}, \mathbf{e})$ and $(\mathbf{s}, \boldsymbol{\gamma})$ remain constitutively uncoupled.

For future use, introduce the constitutive matrix $\mathbf{C}$ that relates $\boldsymbol{\tau}$ to $\boldsymbol{\gamma}$:

$$\tau = \mathbf{C}\gamma, \quad \mathbf{C} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{bmatrix}. \tag{19}$$

2.3. *Reduction to classical elasticity*

Micropolar elasticity reduces to classical linear elasticity if the coupled body force $\mathbf{c}$ vanishes. If so, the second equilibrium equation $2\mathbf{s}+\mathbf{c} = \mathbf{0}$ shows that $\mathbf{s} = \mathbf{0}$, and $\tau = \sigma + \mathbf{s} = \sigma$ is symmetric. Under the assumption that $\mathbf{G}$ is non-singular, the second constitutive equation in (16) gives $\phi = \mathbf{G}^{-1}\mathbf{s} = \mathbf{0}$, and $\gamma = \mathbf{e}+\phi = \mathbf{e}$ is symmetric. Furthermore, $\theta = \omega$, that is, microrotations and continuum-mechanics rotations coalesce.

2.4. *Field dependency*

For the investigation of variational methods in Sections 3 and 4, the field-dependency notational conventions used by Felippa (1989a, b, c, 1992) and Felippa and Militello (1989, 1990) are followed. An *independently varied* field will be identified by a superposed tilde, for example $\tilde{\mathbf{u}}$. A dependent field is identified by writing the independent field symbol as a superscript. For example, if the displacements are independently varied, the derived symmetric strain and stress fields are

$$\mathbf{e}^u = \mathbf{D}\tilde{\mathbf{u}}, \quad \sigma^u = \mathbf{E}\mathbf{e}^u = \mathbf{E}\mathbf{D}\tilde{\mathbf{u}}. \tag{20}$$

Using this convention, tildeless symbols such as $\mathbf{u}$, $\mathbf{e}$ and σ are reserved for the *exact* or *generic* fields. If a symbol derives from two independently varied fields, both fields appear as superscripts: for example $\phi^{u\theta} = \mathbf{R}\tilde{\mathbf{u}} - \tilde{\theta}$.

2.5. *Integral abbreviations*

Volume and surface integrals may be abbreviated by placing domain-subscripted parentheses and brackets, respectively, around the integrand. For example:

$$(f)_V \overset{\text{def}}{=} \int_V f\,\mathrm{d}V, \quad [f]_S \overset{\text{def}}{=} \int_S f\,\mathrm{d}S, \quad [f]_{S_d} \overset{\text{def}}{=} \int_{S_d} f\,\mathrm{d}S, \quad [f]_{S_t} \overset{\text{def}}{=} \int_{S_t} f\,\mathrm{d}S. \tag{21}$$

If $\mathbf{f}$ and $\mathbf{g}$ are vector functions, and $\mathbf{p}$ and $\mathbf{q}$ tensor functions, their inner product over V is denoted in the usual manner:

$$(\mathbf{f}, \mathbf{g})_V \overset{\text{def}}{=} \int_V f_i g_i \,\mathrm{d}V = \int_V \mathbf{f}^T\mathbf{g}\,\mathrm{d}V, \quad (\mathbf{p}, \mathbf{q})_V \overset{\text{def}}{=} \int_V p_{ij}q_{ij}\,\mathrm{d}V = \int_V \mathbf{p}^T\mathbf{q}\,\mathrm{d}V, \tag{22}$$

and similarly for surface integrals, in which case brackets are used.

3. GENERALIZED STRAIN ENERGY FOR CLASSICAL ELASTICITY

The method used to construct parametrized micropolar variational principles in Section 4 represents a generalization of the corresponding principles of classical linear hyperelasticity, which are summarized in this section. These principles have the general form

$$\Pi = U - P. \tag{23}$$

Here U is the generalized strain energy, which characterizes the stored energy of deformation and P is the forcing potential, which characterizes all other contributions. The conventional form of P is

$$P^c = (\mathbf{b}, \tilde{\mathbf{u}})_V + [\tilde{\mathbf{u}} - \hat{\mathbf{d}}, \tilde{\sigma}_n]_{S_d} + [\hat{\mathbf{t}}, \tilde{\mathbf{u}}]_{S_t}, \tag{24}$$

where $\sigma_n = \sigma^T\mathbf{n}$, $\mathbf{n}$ being the unit external normal on S. The other two forms of P, called

P^d and P^t for displacement-generalized and traction-generalized, respectively, are studied by Felippa (1989a, b, c). These (mesh-dependent) forms are of interest in hybrid finite element discretizations. As the forcing potential is not affected by parametrization, attention will be focused on U.

For a *compressible* material, the generalized strain energy introduced in Felippa and Militello (1989, 1990) has the following structure:

$$U = \tfrac{1}{2}j_{11}(\tilde{\sigma}, \mathbf{e}^\sigma)_V + j_{12}(\tilde{\sigma}, \tilde{\mathbf{e}})_V + j_{13}(\tilde{\sigma}, \mathbf{e}^u)_V + \tfrac{1}{2}j_{22}(\sigma^e, \tilde{\mathbf{e}})_V + j_{23}(\sigma^e, \mathbf{e}^u)_V + \tfrac{1}{2}j_{33}(\sigma^u, \mathbf{e}^u)_V, \tag{25}$$

where j_{11} through j_{33} are numerical coefficients. The three independent fields are stresses $\tilde{\sigma}$, strains $\tilde{\mathbf{e}}$ and displacements $\tilde{\mathbf{u}}$. Following the matrix notational conventions stated in Section 2.4 the derived fields that appear in (25) are

$$\sigma^e = \mathbf{E}\tilde{\mathbf{e}}, \quad \sigma^u = \mathbf{ED}\tilde{\mathbf{u}}, \quad \mathbf{e}^\sigma = \mathbf{E}^{-1}\tilde{\sigma}, \quad \mathbf{e}^u = \mathbf{D}\tilde{\mathbf{u}}. \tag{26}$$

As an example, the U of Hu–Washizu's functional is obtained by setting $j_{12} = -1$, $j_{13} = 1$, $j_{22} = 1$, all others being zero:

$$U_H(\tilde{\sigma}, \tilde{\mathbf{e}}, \tilde{\mathbf{u}}) = \tfrac{1}{2}(\sigma^e, \tilde{\mathbf{e}})_V + \tfrac{1}{2}(\tilde{\sigma}, \mathbf{e}^u - \tilde{\mathbf{e}})_V + \tfrac{1}{2}(\sigma^u - \sigma^e, \mathbf{e}^\sigma)_V = \tfrac{1}{2}(\sigma^e, \tilde{\mathbf{e}})_V + (\tilde{\sigma}, \mathbf{e}^u - \tilde{\mathbf{e}})_V. \tag{27}$$

Equation (25) can be rewritten in matrix form as

$$U = \frac{1}{2}\int_V \begin{Bmatrix} \tilde{\sigma} \\ \sigma^e \\ \sigma^u \end{Bmatrix}^T \begin{bmatrix} j_{11}\mathbf{I} & j_{12}\mathbf{I} & j_{13}\mathbf{I} \\ & j_{22}\mathbf{I} & j_{23}\mathbf{I} \\ \text{symm.} & & j_{33}\mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{e}^\sigma \\ \tilde{\mathbf{e}} \\ \mathbf{e}^u \end{Bmatrix} \mathrm{d}V, \tag{28}$$

where $\mathbf{I}$ denotes the 6×6 identity matrix. The functional-generating symmetric matrix

$$\mathbf{J} = \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{12} & j_{22} & j_{23} \\ j_{13} & j_{23} & j_{33} \end{bmatrix} \tag{29}$$

is seen to fully characterize (25) and consequently, once the forcing potential P is selected, the functional (23). (To justify the symmetry of $\mathbf{J}$ note, for example, that $j_{13}(\tilde{\sigma}, \mathbf{e}^u)_V = \tfrac{1}{2}j_{13}(\tilde{\sigma}, \mathbf{e}^u)_V + \tfrac{1}{2}j_{13}(\mathbf{e}^\sigma, \sigma^u)_V$, etc.)

On replacing (26) into (28), U may be expressed in terms of the three independent fields as

$$U = \frac{1}{2}\int_V \begin{Bmatrix} \tilde{\sigma} \\ \tilde{\mathbf{e}} \\ \tilde{\mathbf{u}} \end{Bmatrix}^T \begin{bmatrix} j_{11}\mathbf{E}^{-1} & j_{12}\mathbf{I} & j_{13}\mathbf{D} \\ j_{12}\mathbf{I} & j_{22}\mathbf{E} & j_{23}\mathbf{ED} \\ j_{13}\mathbf{D}^T & j_{23}\mathbf{D}^T\mathbf{E} & j_{33}\mathbf{D}^T\mathbf{ED} \end{bmatrix} \begin{Bmatrix} \tilde{\sigma} \\ \tilde{\mathbf{e}} \\ \tilde{\mathbf{u}} \end{Bmatrix} \mathrm{d}V. \tag{30}$$

Using (30) the first variation of U may be presented as

$$\delta U = (\Delta\mathbf{e}, \delta\tilde{\sigma})_V + (\Delta\sigma, \delta\tilde{\mathbf{e}})_V - (\operatorname{div}\sigma', \delta\tilde{\mathbf{u}})_V + [\sigma'_n, \delta\tilde{\mathbf{u}}]_S, \tag{31}$$

where

$$\Delta\mathbf{e} = j_{11}\mathbf{e}^\sigma + j_{12}\tilde{\mathbf{e}} + j_{13}\mathbf{e}^u, \quad \Delta\sigma = j_{12}\tilde{\sigma} + j_{22}\sigma^e + j_{23}\sigma^u, \quad \sigma' = j_{13}\tilde{\sigma} + j_{23}\sigma^e + j_{33}\sigma^u. \tag{32}$$

The last term in (32) combines with contributions from the forcing potential variation.

For example, if P is the conventional forcing potential (24), the complete variation of $\Pi^c = U - P^c$ is

$$\delta\Pi^c = (\Delta\mathbf{e}, \delta\tilde{\boldsymbol{\sigma}})_V + (\Delta\boldsymbol{\sigma}, \delta\tilde{\mathbf{e}})_V - (\mathrm{div}\,\boldsymbol{\sigma}' + \mathbf{b}, \delta\tilde{\mathbf{u}})_V + [\boldsymbol{\sigma}'_n - \hat{\mathbf{t}}, \delta\tilde{\mathbf{u}}]_{S_t} - [\tilde{\mathbf{u}} - \hat{\mathbf{d}}, \delta\tilde{\boldsymbol{\sigma}}_n]_{S_d}. \quad (33)$$

Using P^d or P^t does not change the volume terms. Consequently the Euler equations associated with the volume terms of the first variation

$$\Delta\mathbf{e} = \mathbf{0}, \quad \Delta\boldsymbol{\sigma} = \mathbf{0}, \quad \mathrm{div}\,\boldsymbol{\sigma}' + \mathbf{b} = \mathbf{0} \quad (34)$$

are independent of the forcing potential. For consistency of the Euler equations with the field equations of classical elasticity one must have $\Delta\mathbf{e} = \mathbf{0}$, $\Delta\boldsymbol{\sigma} = \mathbf{0}$ and $\boldsymbol{\sigma}' = \boldsymbol{\sigma}$ if the assumed stress and strain fields reduce to the exact ones. Therefore

$$j_{11} + j_{12} + j_{13} = 0, \quad j_{12} + j_{22} + j_{23} = 0, \quad j_{13} + j_{23} + j_{33} = 1. \quad (35)$$

Because of these constraints, the maximum number of independent parameters that define the entries of matrix $\mathbf{J}$ is three. The specialization of these functionals to conventional and parametrized forms is discussed by Felippa and Militello (1989, 1990).

Insofar as $\mathbf{E}^{-1}$ appears in (30), this development is valid only for compressible elasticity. Extensions of this variational principle to cover incompressibility are discussed by Felippa (1992).

4. GENERALIZED STRAIN ENERGY FOR MICROPOLAR ELASTICITY

For a micropolar elastic material without couple stresses the variational principle is structurally similar to (23):

$$\Pi_m = U_m - P_m, \quad (36)$$

where U_m now also depends on $\tilde{\mathbf{s}}$, $\tilde{\boldsymbol{\phi}}$ and $\tilde{\boldsymbol{\theta}}$, and P_m may be P^c_m, P^d_m or P^t_m. The following generalization of U to U_m is postulated:

$$U_m = \frac{1}{2}\int_V \begin{Bmatrix} \tilde{\boldsymbol{\sigma}} \\ \boldsymbol{\sigma}^e \\ \boldsymbol{\sigma}^u \\ \tilde{\mathbf{s}} \\ \mathbf{s}^{\phi} \\ \mathbf{s}^{u\theta} \end{Bmatrix}^T \begin{bmatrix} j_{11}\mathbf{I}_6 & j_{12}\mathbf{I}_6 & j_{13}\mathbf{I}_6 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ j_{12}\mathbf{I}_6 & j_{22}\mathbf{I}_6 & j_{23}\mathbf{I}_6 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ j_{13}\mathbf{I}_6 & j_{23}\mathbf{I}_6 & j_{33}\mathbf{I}_6 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & j_{44}\mathbf{I}_3 & j_{45}\mathbf{I}_3 & j_{46}\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & j_{45}\mathbf{I}_3 & j_{55}\mathbf{I}_3 & j_{56}\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & j_{46}\mathbf{I}_3 & j_{56}\mathbf{I}_3 & j_{66}\mathbf{I}_3 \end{bmatrix} \begin{Bmatrix} \mathbf{e}^{\sigma} \\ \tilde{\mathbf{e}} \\ \mathbf{e}^u \\ \boldsymbol{\phi}^s \\ \boldsymbol{\phi} \\ \boldsymbol{\phi}^{u\theta} \end{Bmatrix} \mathrm{d}V, \quad (37)$$

where $\mathbf{I}_6$ and $\mathbf{I}_3$ denote the identity matrices of order 6 and 3, respectively, and the new derived fields are

$$\boldsymbol{\phi}^s = \mathbf{G}^{-1}\tilde{\mathbf{s}}, \quad \mathbf{s}^{\phi} = \mathbf{G}\tilde{\boldsymbol{\phi}}, \quad \boldsymbol{\phi}^{u\theta} = \mathbf{R}\tilde{\mathbf{u}} - \tilde{\boldsymbol{\theta}}, \quad \mathbf{s}^{u\theta} = \mathbf{G}\boldsymbol{\phi}^{u\theta} = \mathbf{G}(\mathbf{R}\tilde{\mathbf{u}} - \tilde{\boldsymbol{\theta}}). \quad (38)$$

The block structure of the kernel matrix in (37) results from the inner product orthogonality (14) of symmetric and antisymmetric tensors. The symmetry of the j coefficients is an assumption that remains to be verified.

On substituting (38) and (26) into (37), U_m is expressed in terms of the six independently varied fields $\tilde{\boldsymbol{\sigma}}$, $\tilde{\mathbf{e}}$, $\tilde{\mathbf{u}}$, $\tilde{\mathbf{s}}$, $\tilde{\boldsymbol{\phi}}$ and $\tilde{\boldsymbol{\theta}}$:

$$U_m = \frac{1}{2}\int_V \begin{Bmatrix} \tilde{\boldsymbol\sigma} \\ \tilde{\mathbf{e}} \\ \tilde{\mathbf{u}} \\ \tilde{\mathbf{s}} \\ \tilde{\boldsymbol\phi} \\ \tilde{\boldsymbol\theta} \end{Bmatrix}^T \begin{bmatrix} j_{11}\mathbf{E}^{-1} & j_{12}\mathbf{I}_6 & j_{13}\mathbf{D} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ j_{12}\mathbf{I}_6 & j_{22}\mathbf{E} & j_{23}\mathbf{ED} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ j_{13}\mathbf{D}^T & j_{23}\mathbf{D}^T\mathbf{E} & j_{33}\mathbf{D}^T\mathbf{ED} + j_{66}\mathbf{R}^T\mathbf{GR} & j_{46}\mathbf{R}^T & j_{56}\mathbf{R}^T\mathbf{G} & -j_{66}\mathbf{R}^T\mathbf{G} \\ \mathbf{0} & \mathbf{0} & j_{46}\mathbf{R} & j_{44}\mathbf{G}^{-1} & j_{45}\mathbf{I}_3 & -j_{46}\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & j_{56}\mathbf{GR} & j_{45}\mathbf{I}_3 & j_{55}\mathbf{G} & -j_{56}\mathbf{G} \\ \mathbf{0} & \mathbf{0} & -j_{66}\mathbf{GR} & -j_{46}\mathbf{I}_3 & -j_{56}\mathbf{G} & j_{66}\mathbf{G} \end{bmatrix} \begin{Bmatrix} \tilde{\boldsymbol\sigma} \\ \tilde{\mathbf{e}} \\ \tilde{\mathbf{u}} \\ \tilde{\mathbf{s}} \\ \tilde{\boldsymbol\phi} \\ \tilde{\boldsymbol\theta} \end{Bmatrix} \mathrm{d}V. \tag{39}$$

The kernel matrix in the above quadratic form must be symmetric, a condition that verifies the symmetry assumptions in (37). As for the forcing poential, the conventional form changes to

$$P_m^c = (\mathbf{b}, \tilde{\mathbf{u}})_V + \tfrac{1}{2}(\mathbf{c}, \tilde{\boldsymbol\theta})_V + [\tilde{\mathbf{u}} - \hat{\mathbf{d}}, \boldsymbol\tau_n]_{S_d} + [\hat{\mathbf{t}}, \tilde{\mathbf{u}}]_{S_t} = P^c + \tfrac{1}{2}(\mathbf{c}, \boldsymbol\theta)_V + [\tilde{\mathbf{u}} - \hat{\mathbf{d}}, \mathbf{s}_n]_{S_d}. \tag{40}$$

Similarly, the generalized forcing potentials P_m^d and P_m^t are obtained by augmenting P^d and P^t, respectively, with $\frac{1}{2}(\mathbf{c}, \tilde{\boldsymbol\theta})_V + [\tilde{\mathbf{u}} - \hat{\mathbf{d}}, \mathbf{s}]S_d$. [The $\frac{1}{2}$ in the $\mathbf{c}$ term arises from the presence of factor 2 in the definition (9) of the microrotation vector $\boldsymbol\theta$.]

The first variation of U_m is

$$\begin{aligned}\delta U_m = {} & (\Delta\mathbf{e}, \delta\tilde{\boldsymbol\sigma})_V + (\Delta\boldsymbol\sigma, \delta\tilde{\mathbf{e}})_V - (\mathbf{D}^T\boldsymbol\sigma' + \mathbf{R}^T\mathbf{s}', \delta\tilde{\mathbf{u}})_V \\ & + (\Delta\boldsymbol\phi, \delta\tilde{\mathbf{s}})_V + (\Delta\mathbf{s}, \delta\tilde{\boldsymbol\phi})_V - (\mathbf{s}', \delta\tilde{\boldsymbol\theta})_V + [\boldsymbol\sigma_n' + \mathbf{s}_n', \delta\tilde{\mathbf{u}}_n]_S, \end{aligned} \tag{41}$$

where $\Delta\mathbf{e}$, $\Delta\boldsymbol\sigma$ and $\boldsymbol\sigma$ are the same as in (32), and

$$\Delta\boldsymbol\phi = j_{44}\boldsymbol\phi^s + j_{45}\tilde{\boldsymbol\phi} + j_{46}\boldsymbol\phi^{u\theta}, \quad \Delta\mathbf{s} = j_{45}\tilde{\mathbf{s}} + j_{55}\mathbf{s}^{\phi} + j_{56}\mathbf{s}^{u\theta}, \quad \mathbf{s}' = j_{46}\tilde{\mathbf{s}} + j_{56}\mathbf{s}^{\phi} + j_{66}\mathbf{s}^{u\theta}. \tag{42}$$

Note that $(\mathbf{D}^T\boldsymbol\sigma' + \mathbf{R}^T\mathbf{s}') = \operatorname{div}\boldsymbol\sigma' + \operatorname{div}\mathbf{s}' = \operatorname{div}\boldsymbol\tau'$, where $\boldsymbol\tau' = {}^*\boldsymbol\sigma' + {}^*\mathbf{s}'$. The first variation of $\Pi_m = U_m - P_m^c$ is

$$\begin{aligned}\delta\Pi_m = {} & (\Delta\mathbf{e}, \delta\tilde{\boldsymbol\sigma})_V + (\Delta\boldsymbol\sigma, \delta\tilde{\mathbf{e}})_V - (\operatorname{div}\boldsymbol\tau' + \mathbf{b}, \delta\tilde{\mathbf{u}})_V + (\Delta\boldsymbol\phi, \delta\tilde{\mathbf{s}})_V \\ & + (\Delta\mathbf{s}, \delta\tilde{\boldsymbol\phi})_V - \tfrac{1}{2}(2\mathbf{s}' + \mathbf{c}, \delta\tilde{\boldsymbol\theta})_V + [\boldsymbol\tau_n' - \hat{\mathbf{t}}, \delta\tilde{\mathbf{u}}]_{S_t} - [\tilde{\mathbf{u}} - \hat{\mathbf{d}}, \delta\tilde{\boldsymbol\tau}_n]_{S_d}. \end{aligned} \tag{43}$$

Following the same argument as in Section 3, it is found that consistency with the field equations requires, in addition to (35), that

$$j_{44} + j_{45} + j_{46} = 0, \quad j_{45} + j_{55} + j_{56} = 0, \quad j_{46} + j_{56} + j_{66} = 1. \tag{44}$$

It follows that the parametrized functional of micropolar elasticity

$$\Pi_m = U_m(\tilde{\boldsymbol\sigma}, \tilde{\mathbf{e}}, \tilde{\mathbf{u}}, \tilde{\mathbf{s}}, \tilde{\boldsymbol\phi}, \tilde{\boldsymbol\theta}) - P_m, \tag{45}$$

depends on $12 - 6 = 6$ free parameters through U_m. Specific instances of (45) are characterized by the functional-generating symmetric matrix

$$\mathbf{J}_m = \begin{bmatrix} j_{11} & j_{12} & j_{13} & 0 & 0 & 0 \\ j_{12} & j_{22} & j_{23} & 0 & 0 & 0 \\ j_{13} & j_{23} & j_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & j_{44} & j_{45} & j_{46} \\ 0 & 0 & 0 & j_{45} & j_{55} & j_{56} \\ 0 & 0 & 0 & j_{46} & j_{56} & j_{66} \end{bmatrix}, \tag{46}$$

subjected to the six constraints (35) and (44). The non-zero 3×3 blocks in $\mathbf{J}_m$ characterize weightings for symmetric and antisymmetric fields, respectively, and one is free to "mix or match". For example,

$$\mathbf{J}_m = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \tag{47}$$

represents the choice of the Hu–Washizu principle for both symmetric and antisymmetric fields.

The variational principles of Reissner (1965) and Hughes and Brezzi (1989) will be now examined in light of the preceding developments.

5. NON-POLAR FUNCTIONALS WITH INDEPENDENT ROTATIONS

5.1. *The Reissner functionals*

Reissner (1965) proposed a functional of Hellinger–Reissner type for classical (non-polar) elasticity ($\mathbf{c} = \mathbf{0}$) in which $\mathbf{u}$, $\boldsymbol{\tau}$ and $\boldsymbol{\theta}$ are to be treated as independent fields. In this functional the stress symmetry condition $\mathbf{s} = \mathbf{0}$ appears as a weak condition with $\boldsymbol{\theta}$ playing the rôle of multiplier. In the present notation the functional, herein called $\Pi_{R1} = U_{R1} - P^c_R$, can be written as

$$U_{R1} = -\tfrac{1}{2}(\tilde{\sigma}, \mathbf{E}^{-1}\tilde{\sigma})_V + (\tilde{\tau}, \nabla\tilde{\mathbf{u}} - \tilde{\theta})_V, \quad P^c_R = P^c + [\tilde{\mathbf{u}} - \hat{\mathbf{d}}, \tilde{\mathbf{s}}_n]_{S_d}, \tag{48}$$

where $\nabla\mathbf{u}$ is the gradient of the displacement vector. Expanding inner products, noting that $\boldsymbol{\tau}^T(\nabla\mathbf{u} - \boldsymbol{\theta}) = \boldsymbol{\tau}^T\gamma^{u,\phi} = ({}^*\sigma + {}^*\mathbf{s})^T({}^*\mathbf{e}^u + {}^*\phi^{u,\phi})$, and making use of (13) yields

$$\begin{aligned} U_{R1} &= -\tfrac{1}{2}(\tilde{\sigma}, \mathbf{e}^\sigma)_V + (\tilde{\sigma}, \mathbf{e}^u)_V + (\tilde{\mathbf{s}}, \phi^{u\theta})_V \\ &= -\tfrac{1}{2}(\tilde{\sigma}, \mathbf{e}^\sigma)_V + \tfrac{1}{2}(\tilde{\sigma}, \mathbf{e}^u)_V + \tfrac{1}{2}(\tilde{\sigma}^u, \tilde{\mathbf{e}})_V + \tfrac{1}{2}(\tilde{\mathbf{s}}, \phi^{u\theta})_V + \tfrac{1}{2}(\mathbf{s}^{u\theta}, \tilde{\phi})_V. \end{aligned} \tag{49}$$

This corresponds to taking

$$\mathbf{J}_m = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{50}$$

It can be seen that the first consistency condition in (44), namely $j_{44} + j_{45} + j_{46} = 0$, is violated.

Consequently Π_{R1} is not a valid functional for micropolar elasticity. Inspection of (50) reveals that conditions (44) can be met by simple changing j_{44} to -1, and that is precisely the regularization of Hughes–Brezzi described in Section 5.2.

Reissner also proposed a second functional $\Pi_{R2} = U_{R2} - P_R^c$ of Hu–Washizu type, in which

$$\begin{aligned} U_{R2} &= \tfrac{1}{2}(\tilde{\mathbf{e}}, \mathbf{E}\tilde{\mathbf{e}})_V + (\tilde{\boldsymbol{\sigma}}, \mathbf{e}^u - \tilde{\boldsymbol{\gamma}})_V + (\tilde{\mathbf{s}}, \boldsymbol{\phi}^{u\theta} - \tilde{\boldsymbol{\phi}})_V \\ &= \tfrac{1}{2}(\boldsymbol{\sigma}^e, \tilde{\mathbf{e}}) + \tfrac{1}{2}(\tilde{\boldsymbol{\sigma}}, \mathbf{e}^u - \tilde{\mathbf{e}})_V + \tfrac{1}{2}(\boldsymbol{\sigma}^u - \boldsymbol{\sigma}^e, \mathbf{e}^\sigma)_V + \tfrac{1}{2}(\tilde{\mathbf{s}}, \boldsymbol{\phi}^{u\theta} - \tilde{\boldsymbol{\phi}})_V + \tfrac{1}{2}(\mathbf{s}^{u\theta} - \tilde{\mathbf{s}}^\phi, \boldsymbol{\phi}^s)_V, \end{aligned} \tag{51}$$

which corresponds to the $\mathbf{J}_m$ of (47) except that $j_{55} = 0$. Now the second consistency equation in (44) is violated. Thus this second functional is also inconsistent with micropolar elasticity, but may be corrected by changing j_{55} to 1.

5.2. *The Hughes–Brezzi functionals*

Hughes and Brezzi (1989) investigated the possible application of the Reissner functionals to construct finite elements with "drilling" degrees of freedom for classical elasticity. Their analysis shows that the first Reissner functional would lead to *unstable* discrete approximations. The physical cause of this instability is that deviations from stress symmetry do not produce strain energy. To circumvent that difficulty, they proposed stabilizing U_{R1} by adding a penalty-like term of the form

$$-\frac{1}{2\bar{\kappa}}(\tilde{\mathbf{s}}, \tilde{\mathbf{s}})_V, \tag{52}$$

where $\bar{\kappa} > 0$ is a pseudo-modulus with dimensions of stress (in their paper this modulus is called γ, a symbol used here for total strain). Although $\bar{\kappa}$ plays the same role as κ in the micropolar theory, for the intended application it is a *fictitious* quantity to be chosen by numerical experiments. The term (52) can be encompassed in the present framework by choosing $\mathbf{G} = \bar{\kappa}\mathbf{I}_3$, which allows that term to be written as $-\tfrac{1}{2}(\tilde{\mathbf{s}}, \boldsymbol{\phi}^s)_V$. Adding this to U_{R1} yields the first Hughes–Brezzi functional:

$$\begin{aligned} U_{HB1} &= -\tfrac{1}{2}(\tilde{\boldsymbol{\tau}}, \mathbf{C}^{-1}\tilde{\boldsymbol{\tau}})_V + (\tilde{\boldsymbol{\tau}}, \nabla\tilde{\mathbf{u}} - \tilde{\boldsymbol{\theta}})_V \\ &= -\tfrac{1}{2}(\tilde{\boldsymbol{\sigma}}, \mathbf{e}^\sigma)_V - \tfrac{1}{2}(\tilde{\mathbf{s}}, \boldsymbol{\phi}^s)_V + \tfrac{1}{2}(\tilde{\boldsymbol{\sigma}}, \mathbf{e}^u)_V + \tfrac{1}{2}(\boldsymbol{\sigma}^u, \tilde{\mathbf{e}})_V + \tfrac{1}{2}(\tilde{\mathbf{s}}, \boldsymbol{\phi}^{u\theta})_V + \tfrac{1}{2}(\mathbf{s}^{u\theta}, \tilde{\boldsymbol{\phi}})_V. \end{aligned} \tag{53}$$

This befits the form (37) with the generating matrix

$$\mathbf{J}_m = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \tag{54}$$

whose coefficients satisfy (35) and (44). Thus the stabilization procedure has also the effect of rendering the functional consistent with micropolar elasticity.

For the second Reissner functional, the stabilization term added to U_{R2} is $\tfrac{1}{2}(\mathbf{s}^\phi, \tilde{\boldsymbol{\phi}})_V$, which effectively transforms the first term in (51) from $(\tilde{\mathbf{e}}, \mathbf{E}\tilde{\mathbf{e}})_V$ to $(\tilde{\boldsymbol{\gamma}}, \mathbf{C}\tilde{\boldsymbol{\gamma}})_V$. The resulting $\mathbf{J}_m$ is (47).

An obvious generalization of this "repeating block" rule is

$$\mathbf{J}_m = \begin{bmatrix} j_{11} & j_{12} & j_{13} & 0 & 0 & 0 \\ j_{12} & j_{22} & j_{23} & 0 & 0 & 0 \\ j_{13} & j_{23} & j_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & j_{11} & j_{12} & j_{13} \\ 0 & 0 & 0 & j_{12} & j_{22} & j_{23} \\ 0 & 0 & 0 & j_{13} & j_{23} & j_{33} \end{bmatrix}, \tag{55}$$

with the coefficients satisfying (35). This three-parameter family permits symmetric and antisymmetric stress and strain fields to be *merged* into total stresses and strains. The resulting functionals $\Pi(\tilde{\tau}, \tilde{\gamma}, \tilde{u}, \tilde{\theta})$ may be viewed as having at most four independent fields. Note, however, that this choice is but a special case of (46).

5.3. *A two-field functional*

The simplest generating matrix with the block structure (55) is

$$\mathbf{J}_m = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{56}$$

The resulting two-field functional is $\Pi_A = U_A - P^c$, with

$$U_A(\tilde{u}, \tilde{\theta}) = \tfrac{1}{2}(\bar{\sigma}^u, e^u)_V + \tfrac{1}{2}(s^{u\theta}, \phi^{u\theta})_V. \tag{57}$$

This may be viewed as a generalization of the minimum potential energy functional, to which it reduces if the second term is dropped. It can be obtained from a more general functional for elastoplasticity proposed by Atluri (1980), who recommends taking $\bar{\kappa} = 4\mu$ in $s^{u\theta} = \bar{\kappa}\phi^{u\theta}$. Hughes and Brezzi (1989) also investigated the functional (57) but made no recommendation on $\bar{\kappa}$.

6. CONCLUSIONS

The functional $\Pi_m = U_m - P_m$ extends the parametrized functional $\Pi = U - P$ of classical linear hyperelasticity to include three more independently varied antisymmetric fields: skew stresses, skew strains and microrotations. This extension is made here in the context of micropolar elasticity without couple stresses.

Another application of these functionals is the construction of finite element interpolations for classical linear elasticity in which the rotational field θ is varied independently from the displacements. The objective is to relax stress symmetry into a weak condition. It is in this context that the functionals of Hughes–Brezzi have been proposed. A membrane element with drilling freedoms based on these functionals has recently been constructed by Ibrahimbegovic (1990). The present study indicates that the Hughes–Brezzi functionals fit the framework of micropolar elasticity if fictitious modulus $\bar{\kappa}$ is identified with the micropolar modulus κ.

The Hughes–Brezzi functionals can be readily generalized into a three-parameter family defined by (55), in which the same weighting is applied to symmetric and antisymmetric fields. However this is just a subspace of the six-parameter functional (45) characterized by the $\mathbf{J}_m$ matrix (46), which allows such weights to be separately chosen.

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APPENDIX: PARAMETRIZED FUNCTIONAL FOR A MICROPOLAR MEDIUM WITH COUPLE STRESSES

In this Appendix the preceding variational formulation is extended to account for the presence of couple stresses m_{ij}. Two changes in the field equations occur. The angular-momentum equilibrium equation gains a divergence term:

$$m_{ij,j} + e_{ijk}\sigma_{jk} + c_i = 0. \tag{A1}$$

The constitutive equations must be augmented by a relation between the couple stresses and microrotation vector derivatives, which for the isotropic case is

$$m_{ij} = \pi_1 \delta_{ij}\theta_{k,k} + \pi_2 \theta_{i,j} + \pi_3 \theta_{j,i}. \tag{A2}$$

Here π_1, π_2 and π_3 are constitutive coefficients with dimension of force, and for compactness we have used the microrotational vector components $\theta_1 = 2\theta_{23}$, $\theta_2 = 2\theta_{31}$ and $\theta_3 = 2\theta_{12}$ in accordance to the convention of eqn (9). The gradients of θ_i will be denoted by $\chi_{ij} = \theta_{i,j}$, which may be interpreted as "curvatures".

In addition, the boundary conditions (4)–(5) are augmented with

$$m_{ij}n_j = m_{ni} = \hat{\omega} \text{ on } S_m, \quad \theta_i = \hat{\theta}_i \text{ on } S_\theta, \tag{A3}$$

where $S: S_m \cup S_\theta$.

Next, define the vectors and matrices

$$\mathbf{m} = \{m_{11} \quad m_{22} \quad m_{33} \quad m_{23} \quad m_{31} \quad m_{12} \quad m_{32} \quad m_{13} \quad m_{21}\}^T,$$

$$\chi = \{\chi_{11} \quad \chi_{22} \quad \chi_{33} \quad \chi_{23} \quad \chi_{31} \quad \chi_{12} \quad \chi_{32} \quad \chi_{13} \quad \chi_{21}\}^T,$$

$$\mathbf{H} = \begin{bmatrix} \pi_4 & \pi_1 & \pi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \pi_1 & \pi_4 & \pi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \pi_1 & \pi_1 & \pi_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi_2 & 0 & 0 & \pi_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi_2 & 0 & 0 & \pi_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi_2 & 0 & 0 & \pi_3 \\ 0 & 0 & 0 & \pi_3 & 0 & 0 & \pi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi_3 & 0 & 0 & \pi_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi_3 & 0 & 0 & \pi_2 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \partial/\partial x_1 & 0 & 0 \\ 0 & \partial/\partial x_2 & 0 \\ 0 & 0 & \partial/\partial x_3 \\ 0 & 0 & \partial/\partial x_2 \\ \partial/\partial x_3 & 0 & 0 \\ 0 & \partial/\partial x_1 & 0 \\ 0 & \partial/\partial x_3 & 0 \\ 0 & 0 & \partial/\partial x_1 \\ \partial/\partial x_2 & 0 & 0 \end{bmatrix}, \tag{A4}$$

in which $\pi_4 = \pi_1 + \pi_2 + \pi_3$. Matrix **H** can be generalized to account for anisotropy without difficulty. Little is known experimentally about couple stress constitutive behavior, however, even in the isotropic case.

With the foregoing definitions, the matrix field equations that include the effect of the couple stresses are

$$\boldsymbol{\chi} = \mathbf{Q}\boldsymbol{\theta}, \quad \mathbf{m} = \mathbf{H}\boldsymbol{\chi}, \quad \mathbf{Q}^{T}\mathbf{m} + 2\mathbf{s} + \mathbf{c} = \mathbf{0}. \tag{A5}$$

The first two are appended to the kinematic relations (14) and constitutive equations (15), respectively, whereas the latter replaces the second of (16).

A parametrized variational principle that accounts for couple stresses is easily obtained by including two independently varied fields: couple stresses $\tilde{m}$ and curvatures $\tilde{\chi}$. Functionals U_m and P^c_m are augmented with couple stress terms

$$U_{mcs} = U_m + U_{cs}, \quad P^c_{mcs} = P^c_m + P^c_{cs}, \tag{A6}$$

where

$$U_{cs} = \frac{1}{2}\int_V \begin{Bmatrix} \tilde{\mathbf{m}} \\ \mathbf{m}^{\chi} \\ \mathbf{m}^{\theta} \end{Bmatrix}^{T} \begin{bmatrix} j_{77}\mathbf{I}_9 & j_{78}\mathbf{I}_9 & j_{79}\mathbf{I}_9 \\ & j_{88}\mathbf{I}_9 & j_{89}\mathbf{I}_9 \\ \text{symm.} & & j_{99}\mathbf{I}_9 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\chi}^{m} \\ \tilde{\boldsymbol{\chi}} \\ \boldsymbol{\chi}^{\theta} \end{Bmatrix} \mathrm{d}V, \tag{A7}$$

$$\mathbf{P}^c_{cs} = [\hat{\mathbf{w}}, \tilde{\boldsymbol{\theta}}]_{S_m} + [\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}, \tilde{\mathbf{m}}_n]_{S_\theta}. \tag{A8}$$

The derived fields in (A7) are $\mathbf{m}^{\chi} = \mathbf{H}\tilde{\boldsymbol{\chi}}$, $\boldsymbol{\chi}^{m} = \mathbf{H}^{-1}\mathbf{m}$, $\boldsymbol{\chi}^{\theta} = \mathbf{Q}\boldsymbol{\theta}$ and $\mathbf{m}^{\theta} = \mathbf{H}\mathbf{Q}\boldsymbol{\theta}$; also $\mathbf{I}_9$ denotes the 9×9 identity matrix.

The first variation of $\Pi_{mcs} = U_{mcs} + P^c_{mcs}$ is

$$\begin{aligned} \delta\Pi_{mcs} = {} & (\Delta\mathbf{e}, \delta\tilde{\boldsymbol{\sigma}})_V + (\Delta\boldsymbol{\sigma}, \delta\tilde{\mathbf{e}})_V - (\mathbf{R}^{T}\boldsymbol{\tau}' + \mathbf{b}, \delta\tilde{\mathbf{u}})_V + (\Delta\boldsymbol{\phi}, \delta\tilde{\mathbf{s}})_V \\ & + (\Delta\mathbf{s}, \delta\tilde{\boldsymbol{\phi}})_V - \tfrac{1}{2}(\mathbf{Q}^{T}\mathbf{m}' + 2\mathbf{s}' + \mathbf{c}, \delta\tilde{\boldsymbol{\theta}})_V + [\boldsymbol{\tau}'_n - \hat{\mathbf{t}}, \delta\tilde{\mathbf{u}}]_{S_t} \\ & - [\tilde{\mathbf{u}} - \hat{\mathbf{d}}, \delta\tilde{\boldsymbol{\tau}}_n]_{S_d} + [\mathbf{m}'_n - \hat{\mathbf{w}}, \delta\tilde{\mathbf{u}}]_{S_m} - [\tilde{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}, \delta\tilde{\mathbf{m}}_n]_{S_\theta}, \end{aligned} \tag{A9}$$

where $\mathbf{m}' = j_{79}\tilde{\mathbf{m}} + j_{89}\mathbf{m}^{\chi} + j_{99}\mathbf{m}^{\theta}$. The consistency conditions are (36), (45) and

$$j_{77} + j_{78} + j_{79} = 0, \quad j_{78} + j_{88} + j_{89} = 0, \quad j_{79} + j_{89} + j_{99} = 1. \tag{A10}$$

It is seen that extending the variational principle (45) to accommodate couple stresses brings three additional free parameters, for a total of nine. This may be reduced to three free parameters, however, by extending the rule (55) with another 3×3 repeating block. Note that if one chooses $j_{99} = 1$, others zero, $U_{cs} = \frac{1}{2}(\boldsymbol{\theta}^{T}\mathbf{Q}^{T}\mathbf{H}\mathbf{Q}\boldsymbol{\theta})_V$, and no additional independent fields other than those in (45) appear.

The couple-stress theory of elasticity attracted theoretical attention in the 1960s but it is rarely used in practice, particularly in static situations. For modeling micropolar and oriented media the simpler equations of Section 1 are more common. This is especially true in homogenization of filamentary composite materials, where the body couple c and the micropolar modulus κ can be estimated from component-level non-polar data complemented by statistical and periodicity arguments [see for example, Berglund (1977)].

Although couple stress models can be generated in the continuum limit of regular and defective-lattice theories [see for example, Askar (1985)], the difficulties in characterizing and measuring moduli such as π_1, π_2 and π_3 are significant, and the theory has to be regarded as experimentally inconclusive. Furthermore the additional boundary conditions (A3) are not easily interpreted physically. Consequently the main development of the paper focuses on the zero-couple-stress case. This has the additional advantage that the reduction to the classical non-polar case for finite element development is easily accomplished.